On $\alpha\beta$-Factorization, Special Cases of Simon's Congruence, and Dominos

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GöKi-Workshop

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Introduction

Definition (Subword, Scattered Factor)

Let $v, w \in \Sigma^*$, then $v \preceq w \iff v$ can be embedded in w

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Definition (Simon's Congruence)

For $u, v \in \Sigma^*$ we define $u \sim_k v \quad :\Longleftrightarrow \quad \text{ScatFact}_{\leq k}(u) = \text{ScatFact}_{\leq k}(v)$.

$\alpha\beta$-factorization

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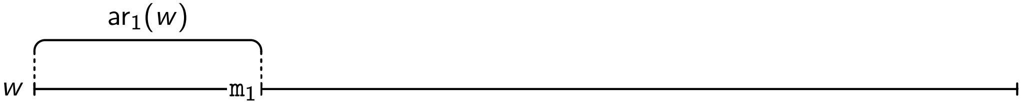

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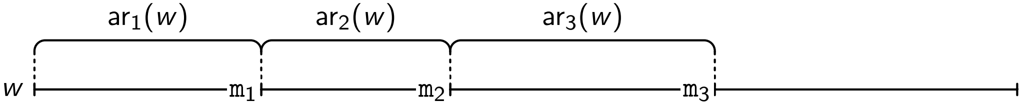

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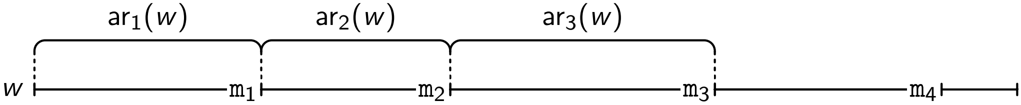

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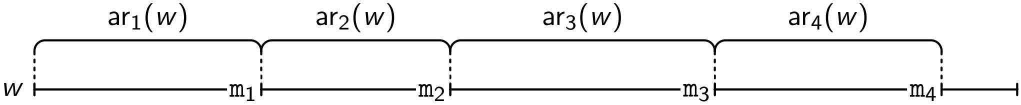

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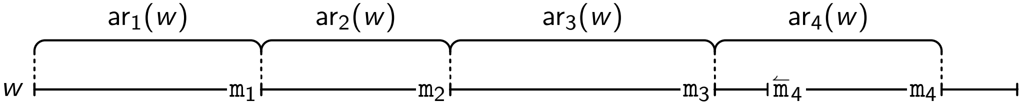

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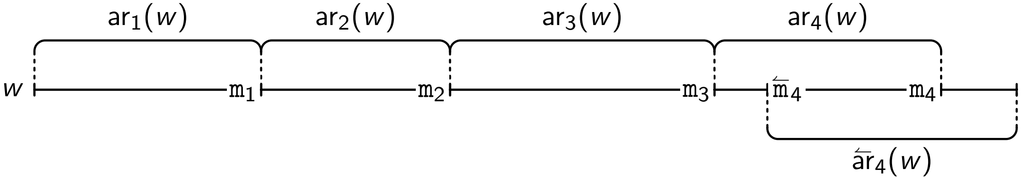

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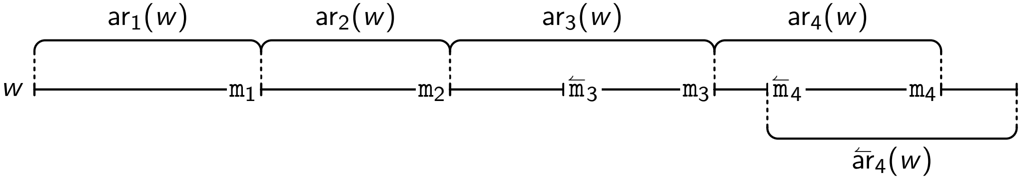

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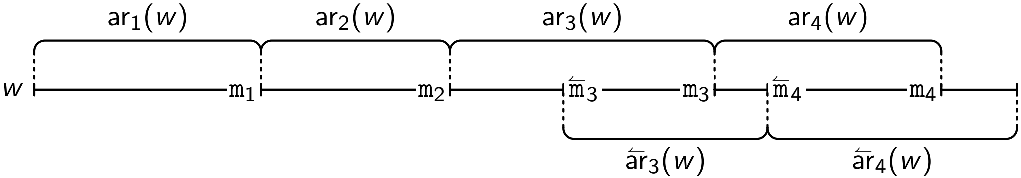

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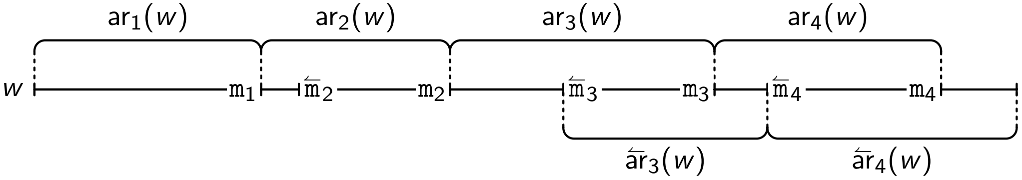

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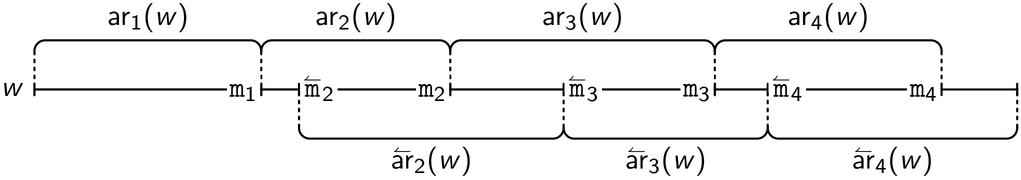

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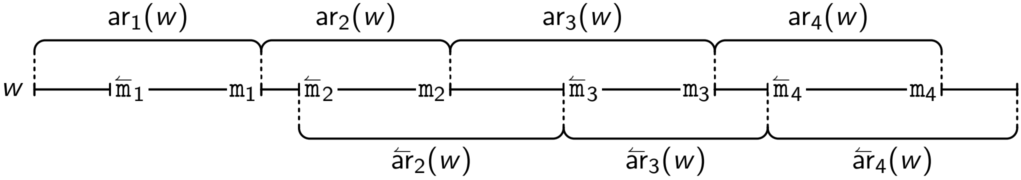

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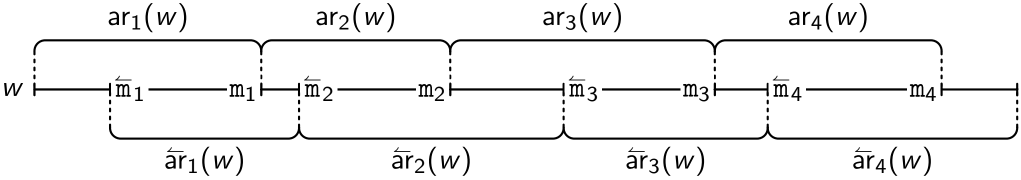

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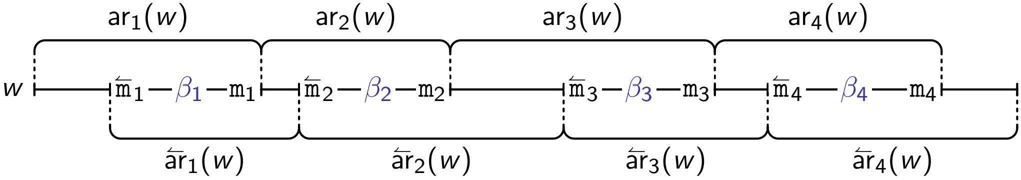

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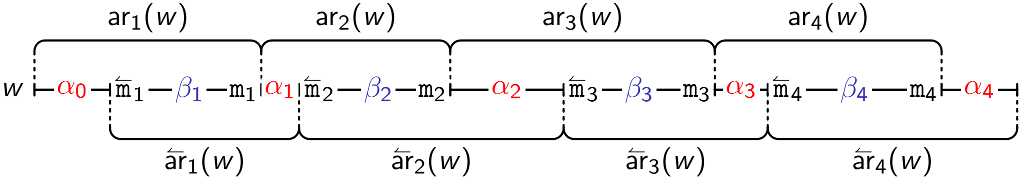

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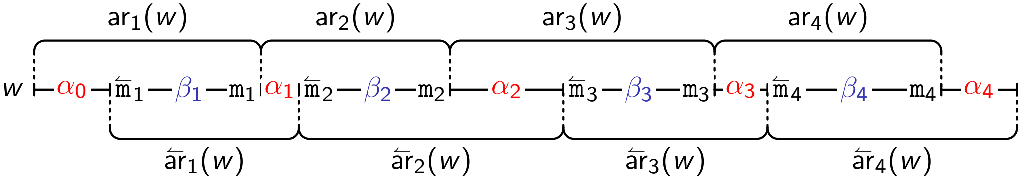

Example abaccabc

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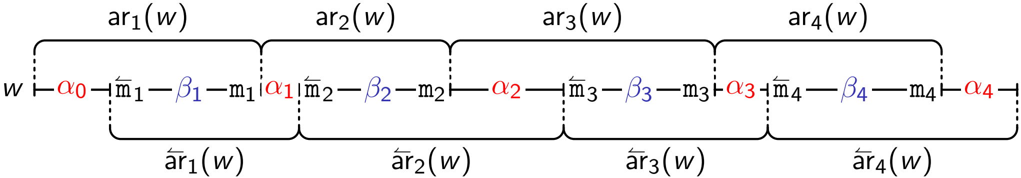

Example abac · cabc

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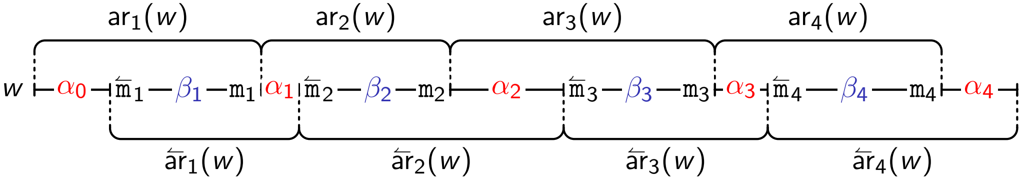

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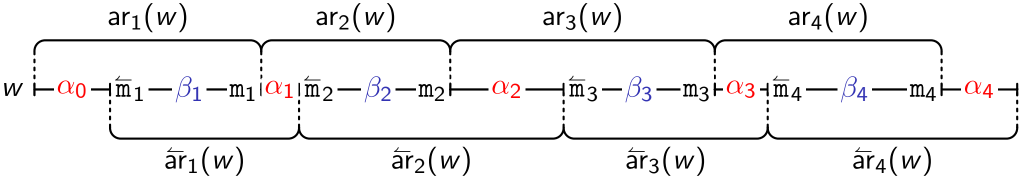

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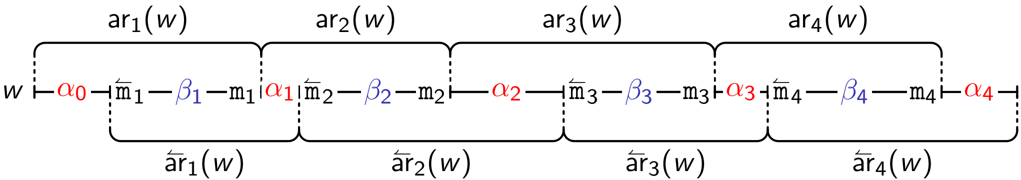

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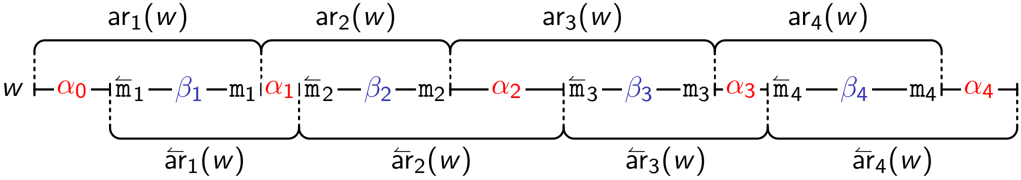

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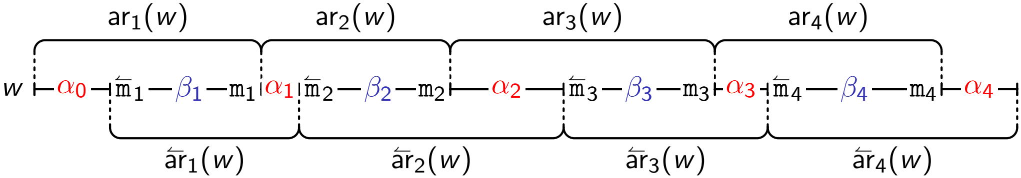

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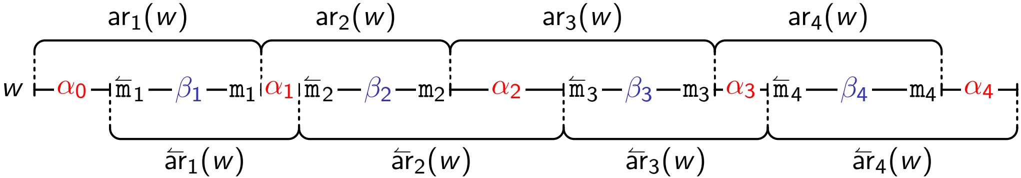

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Universality $\iota(w) := \#\text{arches} \iff \text{max. number s.t. } \Sigma^{\iota(w)} \subseteq \mathrm{ScatFact}(w)$

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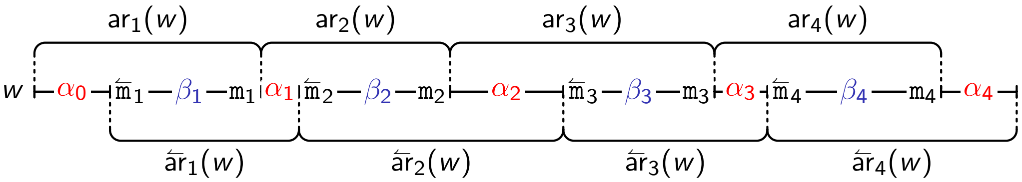

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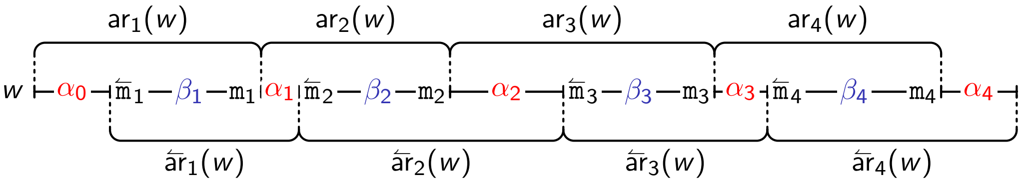

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General Results on $\alpha\beta$-Factorization

Lemma

$u = \alpha_0\beta_1\cdots\alpha_m, v = \alpha'_0\beta'_1\cdots\alpha'_m \in \Sigma^*$ *with* $u \sim_k v$ *and* $m := \iota(u) = \iota(v) < k$, *then*

- $\alpha_i\beta_{i+1}\alpha_{i+1}\cdots\beta_j\alpha_j \sim_{k-m+(j-i)} \alpha'_i\beta'_{i+1}\alpha'_{i+1}\cdots\beta'_j\alpha'_j$ *for* $0 \leq i \leq j \leq m$ *we have*
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Proof Sketch. we show $u \sim_k v \implies \mathrm{ar}_1^{-1}(u) \cdot u \sim_{k-1} \mathrm{ar}_1^{-1}(v) \cdot v$, then induction

$\vdash \mathrm{ar}_1(u) \dashv\vdash \tilde{u} := \mathrm{ar}_1^{-1}(u) \cdot u \dashv \quad u \qquad v \quad \vdash \mathrm{ar}_1(v)\text{-}\mathtt{m}_1 \dashv\vdash \tilde{v} := \mathrm{ar}_1^{-1}(v) \cdot v \dashv$

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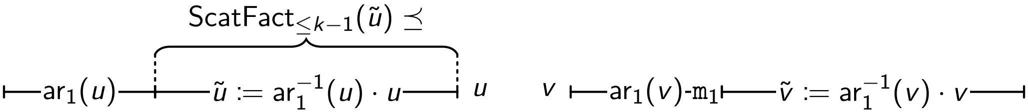

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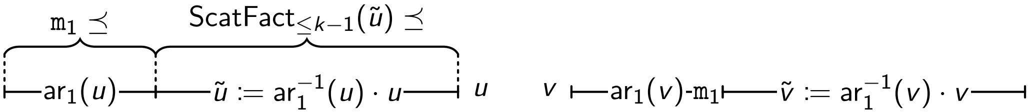

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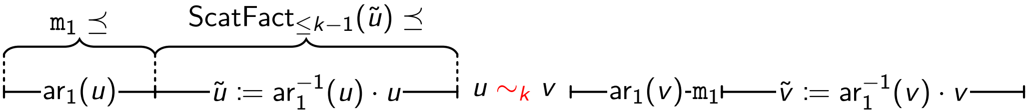

General Results on $\alpha\beta$-Factorization

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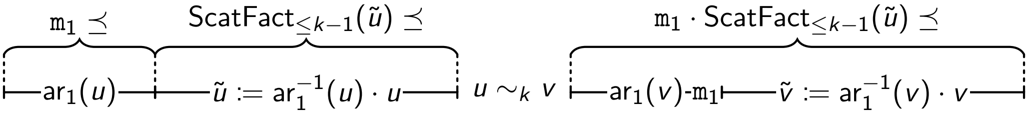

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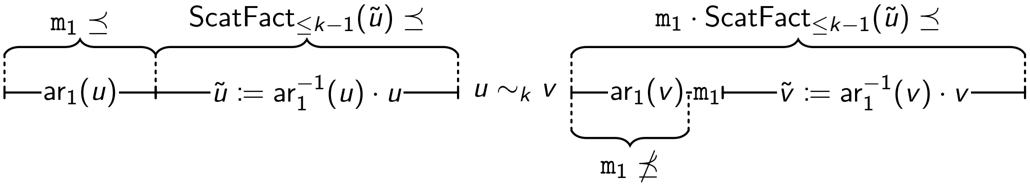

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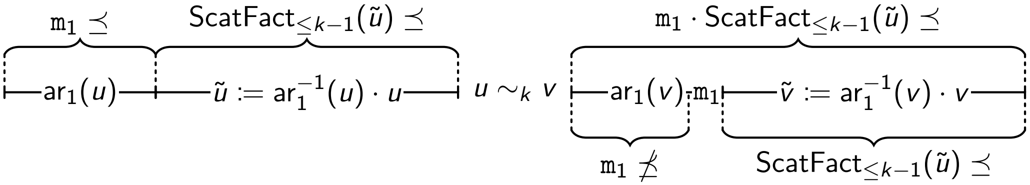

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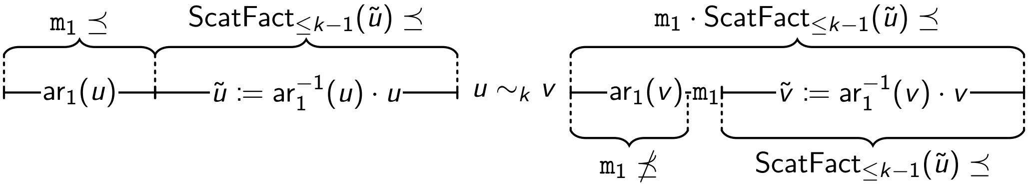

thus $\mathrm{ScatFact}_{\leq k-1}(\tilde{u}) \subseteq \mathrm{ScatFact}_{\leq k-1}(\tilde{v})$ and "$\supseteq$" by symmetry □

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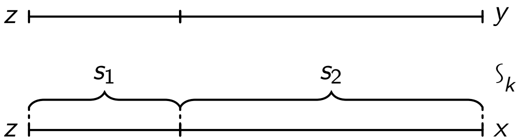

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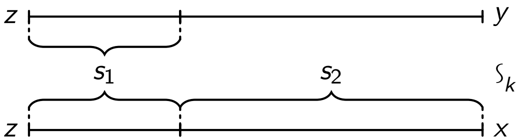

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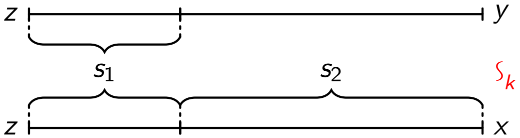

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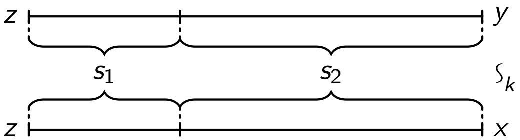

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$u, \tilde{u} \in \Sigma^*$ with $m := \iota(u) = \iota(\tilde{u}) < k$, and $\beta_i = \tilde{\beta}_i$ for all $i \in [m]$, then

$$u \sim_k \tilde{u} \iff \forall i \in [m]_0 . \alpha_i \sim_{k-m} \tilde{\alpha}_i$$

Proof Sketch. We use $x \sim_\ell y \implies xz \sim_{\ell+\iota(z)} yz$ (move at least $\iota(z)$ letters into z):

$$\begin{aligned} u &= \alpha_0\beta_1\alpha_1\beta_2\alpha_2\cdots\beta_{m-1}\alpha_m \\ &\sim_k \tilde{\alpha}_0\beta_1\alpha_1\beta_2\alpha_2\cdots\beta_{m-1}\alpha_m \\ &\sim_k \tilde{\alpha}_0\beta_1\tilde{\alpha}_1\beta_2\alpha_2\cdots\beta_{m-1}\alpha_m \\ &\dots \end{aligned}$$

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– W.l.o.g. we can exchange the α factors to be equal

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General Results on $\alpha\beta$-Factorization

A more general decomposition theorem

Theorem ($\alpha\beta\alpha$-decomposition)

$u, \tilde{u} \in \Sigma^*$ with $m := \iota(u) = \iota(\tilde{u}) < k$, then

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– reduces problem to 1-universal words

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– reduces problem to 1-universal words
– analysis of 1-universal words is hard in general

The Binary Case

- $\beta_i \in \{\mathtt{ab}, \mathtt{ba}, \mathtt{a}, \mathtt{b}\}$ by uniqueness of first and last letter
- $\alpha_i \in \mathtt{a}^* \cup \mathtt{b}^*$ since $\mathrm{alph}(\alpha_i) \subset \Sigma_2$

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Theorem (Binary Characterization)

$u, \tilde{u} \in \Sigma_2^*$ with $m := \iota(u) = \iota(\tilde{u}) < k$, then

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The Binary Case

Finite Classes

Theorem

Let $w \in \Sigma_2^*$, then $|[w]_{\sim_k}| < \infty$ and in particular $|[w]_{\sim_k}| = 1$, if and only if, $\iota(w) < k$ and $|\alpha_i| < k - \iota(w)$ for all $i \in [\iota(w)]_0$.

[4] Lothaire, *Combinatorics on Words*, Corollary 6.2.8.

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- By Simon and Sakarovitch[4] the classes are singletons or infinite
- For words w with $\iota(w) \geq k$ we have $\mathrm{ScatFact}_{\leq k}(w) = \Sigma_2^*$ and thus $w \sim_k w^2$

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- For words w with $\iota(w) \geq k$ we have $\mathrm{ScatFact}_{\leq k}(w) = \Sigma_2^*$ and thus $w \sim_k w^2$
- If there exists $\alpha_i \in \mathtt{x}^{\geq k-\iota(w)}$, then we can freely add more $\mathtt{x}$ to this α_i since $[\alpha_i]_{\sim_{k-\iota(w)}} = \mathtt{x}^{\geq k-\iota(w)}$ □

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The Binary Case

"elementary" MaxSimK

Algorithm 1: MaxSimK for binary words

Input: $u, v \in \Sigma_2^*$

Result: if $u = v$ then ∞ and otherwise the maximum k such that $u \sim_k v$

$(\alpha_0, \beta_1, \ldots, \alpha_{\iota(u)}) := \alpha\beta\text{-Fact}(u)$; // w.r.t. Σ_2

$(\alpha'_0, \beta'_1, \ldots, \alpha'_{\iota(v)}) := \alpha\beta\text{-Fact}(v)$;

if $\iota(u) \neq \iota(v) \vee \text{alph}(u) \neq \text{alph}(v)$ **then** // 2nd condition for $u = \mathtt{x}^i, v = \overline{\mathtt{x}}^j$

  return $\min(\iota(u), \iota(v))$;

else if $\beta_1 = \beta'_1 \wedge \cdots \wedge \beta_{\iota(u)} = \beta'_{\iota(v)}$ **then**

  for $i \in [\iota(u)]_0$ **do** // solve MaxSimK for unary α pairs

    $e_i :=$ **if** $|\alpha_i| = |\alpha'_i|$ **then** ∞ **else** $\min(|\alpha_i|, |\alpha'_i|)$;

  return $\iota(u) + \min\{e_i \mid i \in [\iota(u)]_0\}$;

else

  return $\iota(u)$;

The Binary Case

Enumeration

Idea

- We count classes separated by number of arches
- Extend $\alpha_0\beta_1\cdots\beta_m\alpha_m$ w.r.t $\sim_k$ to $\alpha_0\beta_1\cdots\beta_m\alpha_m\beta_{m+1}\alpha_{m+1}$ w.r.t $\sim_{k+1}$ by choosing suitable β, α
- Choice of β_{m+1} depends only on alph(α_m)

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Calculation

- We count spearated by alphabet of last α
- By Cayley-Hamilton we obtain a reccurence satisfied by the matrix

$$|\{w \in \Sigma_2^* \mid \iota(w) = m\}/\sim_k| = \left\| \begin{pmatrix} k-m & k-m & k-m \\ 1 & 2 & 1 \\ k-m & k-m & k-m \end{pmatrix}^m \cdot \begin{pmatrix} k-m \\ 1 \\ k-m \end{pmatrix} \right\|_1 = c_k^m$$

where $c_k^{-1} := 1$, $c_k^0 := 2k+1$ and $c_k^m := 2\cdot(k-m+1)\cdot c_{k-1}^{m-1} - 2\cdot(k-m)\cdot c_{k-2}^{m-2}$.

The Binary Case

Enumeration

Scat Fact Length		Number of Arches								
		0	1	2	3	4	5	6	7	m
	1	3	1							
	2	5	10	1						
	3	7	26	34	1					
	4	9	50	136	116	1				
	5	11	82	358	712	396	1			
	6	13	122	748	2 564	3 728	1 352	1		
	7	15	170	1 354	6 824	18 364	19 520	4 616	1	
	k	$2k+1$								

The Binary Case

Enumeration

Corollary

Let $k \in \mathbb{N}_0$. Over a binary alphabet, the number of congruence classes of $\sim_k$ is given by

$$|\Sigma_2^*/{\sim_k}| = 1 + \sum_{m=0}^{k-1} c_k^m.$$

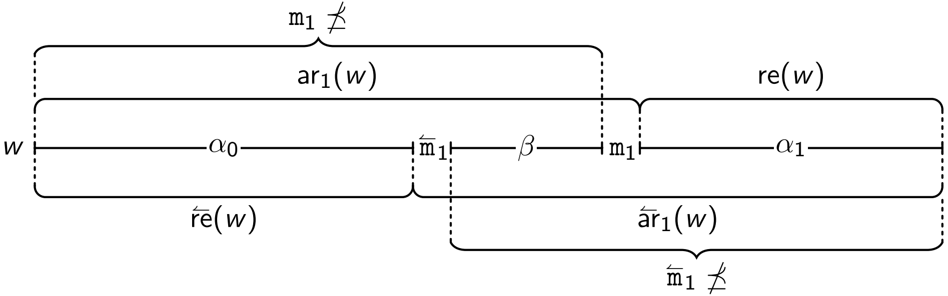
1, 4, 16, 68, 312, 1560, 8528, 50864, 329248, 2298592, 17203264, 137289920, 1162805376, 10409679744, 98146601216, 971532333824, 10068845515264, 108986217860608, 1229343124489216, 14421776248343552, 175640256185075712, 2217046903483447296, 28961312849254387712, 390980190802427064320, ...

Towards the Ternary Case

Observations
- In general $w \sim_k \tilde{w}$ does not imply $\beta_i = \tilde{\beta}_i$
- For example $\mathtt{abab} \cdot \mathtt{ab} \cdot \mathtt{c} \sim_3 \mathtt{abab} \cdot \mathtt{ba} \cdot \mathtt{c}$
- Note that $\alpha_0 = \tilde{\alpha}_0$ is "sufficently universal"

Approach
- By the decomposition theorem, it suffices to characterize $\alpha\beta\alpha$-triples
- α-factors are at most binary, handled by induction

Towards the Ternary Case



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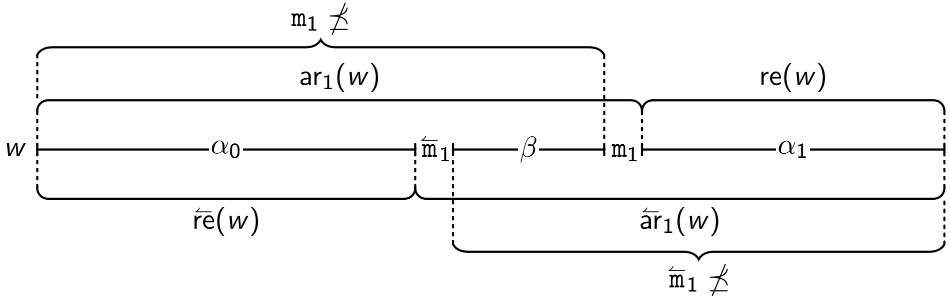

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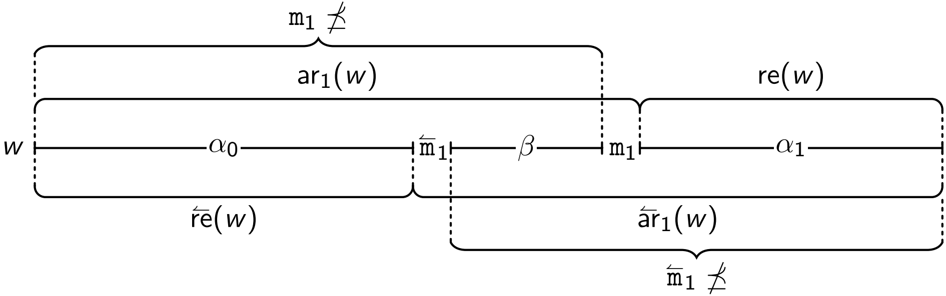

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– Example: $(\mathtt{ababc} \cdot \mathtt{ab} \cdot \mathtt{ab}) \cdot \mathtt{abcd} \cdot \varepsilon \sim_4 (\mathtt{ababc} \cdot \mathtt{ab} \cdot \mathtt{ab}) \cdot \mathtt{bacd} \cdot \varepsilon$

Card	alph(α_0)	alph(α_1)	β RegExp
2–2	$\{a, b\}$	$\{a, c\}$	ba^*c
	$\{a, b\}$	$\{a, b\}$	c
2–1	$\{a, b\}$	$\{c\}$	$(a^+b \mid b^+a)c$
	$\{a, b\}$	$\{a\}$	ba^*c
2–0	$\{a, b\}$	$\emptyset$	$(a^+b \mid b^+a)c$
1–1	$\{a\}$	$\{b\}$	$ab^+c \mid ac^+b \mid ca^+b$
	$\{a\}$	$\{a\}$	ba^*c
1–0	$\{a\}$	$\emptyset$	$ba^*c \mid ab^+c$
0–0	$\emptyset$	$\emptyset$	ab^+c

- $a, b, c \in \Sigma_3$ different; permutation of modi occurs only in case 3 and 5
- exact structure of β determined by $\iota(\alpha_i), \mathrm{alph}(\alpha_i), \mathrm{re}(\alpha_0), \overleftarrow{\mathrm{re}}(\alpha_1)$

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