

(Towards) Poset Type Theory

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TYPES 2024, Copenhagen
June 11, 2024

Motivation

Models of HoTT

Simplicial Sets Model [KL18]

- Closed Types are interpreted by Kan complexes
- Original construction based on Quillen model structure on simplicial sets
- Inherently non-constructive [BC15; BCP15; Par18]

Cubical Models

- Constructive models of HoTT
- Yield nominal extensions of type theory
- Associated type theories have decidability of type checking, normalization
- Unclear connection to homotopy theory

Motivation

Cubical Model Construction

- Models of type theory in presheaves over different cube categories [BCH; CCHM; ABCFHL; OP18; Lic+18; Coq18]
- Crucial ingredients: tiny interval $\mathbb{I} \in \widehat{\square}$, cofibration classifier $\Phi \subseteq \Omega$
- One then defines a notion of *fibrancy structure* $\text{is-fib}_{\Gamma}(A)$ on dep. types $A : \Gamma \rightarrow U$
- Cubical model obtained by *relativizing* presheaf models of ETT

$$\text{Ty}_{\square}(\Gamma) := \prod_{A \in \text{Ty}(\Gamma)} \text{Tm}(\cdot, \text{is-fib}_{\Gamma}(A))$$

- Associated *cubical* type theories (extended syntax + computation rules)

Motivation

Model Structures on Cubical Models

- For these CTTs connection to homotopy theory unclear
- For some cubical models one can (constructively) construct a Quillen model structure from the model of CTT [GS17; Sat17; Awo23]
- Question: Are these Quillen equivalent to $\widehat{\Delta}_{\text{Kan}}$?¹

Model	Equivalent?
Affine (BCH)	No
Cartesian (ABCFHL)	No
Equivariant	Yes
Semilattice	Yes
Dedekind	?
De Morgan (CCHM)	No

Sattler's Poset Model

Construction Sketch

- Recent discovery by Christian Sattler²
- Localization of a cubical model on $\widehat{\mathbf{Pos}}$, where \mathbf{Pos} finite, non-empty posets
- $\Delta_+ \xrightarrow{j} \Delta \xrightarrow{i} \mathbf{Pos}$ induces $M := (ij)_* \circ (ij)^*$ (external) lex operation on $\widehat{\mathbf{Pos}}_{\square}$

$$A \in \text{Ty}(\Gamma), \rho \in \Gamma(I) \quad \text{then} \quad \widetilde{MA}\rho = \left\{ \left\{ u_f \in A\rho f \right\}_{f:[n] \rightarrow I} \mid \begin{array}{l} u_f g = u_{fg} \text{ for} \\ g \in \Delta_+([m], [n]) \end{array} \right\}$$

- $\widehat{\mathbf{Pos}}_M$ has an associated Quillen model structure that is equivalent to $\widehat{\Delta}_{\text{Kan}}$
- Has constructively a lot of desirable properties
- Our goal: Type theory for this model (currently *without* lex operation)

Sattler's Poset Model

Properties

The following hold in $\widehat{\mathbf{Pos}}_M$ (even in a constructive meta theory)
 \implies maybe a (computational) type theory with them is possible

Dependent Choice For every tower of surjections

$$A_0 \ll \xleftarrow{f_0} A_1 \ll \xleftarrow{f_1} A_2 \ll \xleftarrow{f_2} \dots$$

limit is surjective, or if there merely exists $a_0 : A_0$ then there merely exist elements

$$a_1 : A_1, a_2 : A_2, \dots \quad \text{such that} \quad f_i(a_{i+1}) = a_i.$$

Whitehead's Principle Let $f : A \rightarrow B$ a function such that $\|f\|_0 : \|A\|_0 \rightarrow \|B\|_0$ is a bijection, and $\pi_k(f) : \pi_k(A, a) \rightarrow \pi_k(B, f(a))$ is a bijection for all $k \geq 1$ and $a : A$, then f is an equivalence.³

Poset Type Theory

Goal (cubical) type theory with model in $\widehat{\mathbf{Pos}}_M$ with features validated there

Currently Cubical type theory for $\widehat{\mathbf{Pos}}_{\square}$ and $\widehat{\mathbf{Pos}}_M$

- Should be a first step towards type theory with modal types

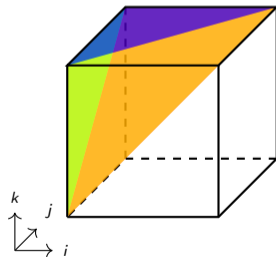
Prototype <https://github.com/JonasHoefer/poset-type-theory>

- Univalence
- Higher Inductive Types

Poset Type Theory

- Cubical type theory with
 1. Connections $\wedge, \vee : \mathbb{I} \times \mathbb{I} \rightarrow \mathbb{I}$
 2. $\text{coe}^{r \rightarrow s}$ and $\text{hcomp}^{r \rightarrow s}$ with arbitrary endpoints
 3. Diagonal cofibrations $r = s : \Phi$ for $r, s : \mathbb{I}$
 } CCTT and Dedekind
CCHM as subtheories
- Contexts can contain interval variables, and equations between *terms*

$$\Gamma, i : \mathbb{I}, j : \mathbb{I}, k : \mathbb{I}, i \vee j = j \wedge k \vdash a : A$$



- Connections allow definition of *equivariant* coe operation

Poset Type Theory

Posets vs Lattices Birkhoff duality: $(\mathbf{Pos}_{\mathbf{Fin}, \neq \emptyset})^{\text{op}} \simeq \mathbf{DLat}_{\mathbf{Fin}, \perp \neq \top}$

- Distributive lattice L is finite iff it is finitely presented $L \cong \langle x_0, x_1, \dots \mid R_0, R_1, \dots \rangle$
- Intuition: for $\rho \in \Gamma(\langle x_0, x_1, \dots \mid R_0, R_1, \dots \rangle)$ the elements of A_ρ are values of A depending on the names x_0, x_1, \dots subject to equations R_0, R_1, \dots
- Birkhoff duality allows reduction of problems needed for evaluation and type checking (e.g. word problem for f.p. distrib lattices) to boolean SAT problems

Pseudo Complement

- In the cubical type theory, we manipulate partial elements of types $[\varphi_1 \hookrightarrow a_1, \dots]$
- The cofibration classifier is pseudo-complemented $-^* : \Phi \rightarrow \Phi$
- Using φ^* , one can avoid hcomps with empty systems, similar to [Ang21]

$$\underline{\text{hcomp}}^{r_0 \rightarrow r_1} A a_0 [\varphi \hookrightarrow \lambda_i . u i] := \text{hcomp}^{r_0 \rightarrow r_1} A a_0 [\varphi \hookrightarrow \lambda_i . u i, \varphi^* \hookrightarrow \lambda_i . a_0]$$

Future Work

- Extension of type theory with additional constructs to reflect $\widehat{\mathbf{Pos}}_M$
 - Full type theory with dependent choice and Whitehead's principle
 - Unclear how to represent modal types in syntax and compute with them
- Extend model construction to sheaf models, e.g., synthetic stone duality

Poset Type Theory

Pseudo Complement

Complement Ω is pseudo-complemented ($\varphi \wedge \varphi^* = 0$)

- Φ is closed under this operation
- $-^* : \Phi \rightarrow \Phi$ is natural (but not stable under reduction)
- complete hcomps by adding φ^* branch

hComps Avoid empty systems (values of the form $\text{hcomp}^{r_0 \rightarrow r_1} A u_0 []$)

- add branch $\varphi^* \hookrightarrow \lambda_z u_0$

$$\text{hcomp}^{r_0 \rightarrow r_1} A u_0 [\varphi \hookrightarrow \lambda_z u, \varphi^* \hookrightarrow \lambda_z u_0]$$

- clearly compatible since $\varphi \wedge \varphi^* = 0$
- $\psi := \varphi \vee \varphi^*$ is dense: $\psi^{**} = 1$
- since $0^* = 1$ and $1^* = 0$, restrictions yield no empty systems

Poset Type Theory

Pseudo Complement

- Reduction of hcomp in Σ is problematic

$$\begin{aligned}
 & \left(\text{hcomp}^{r \rightarrow s} (\Sigma AB) u_0 \begin{bmatrix} \varphi & \hookrightarrow & u \\ \varphi^* & \hookrightarrow & \lambda_z.u_0 \end{bmatrix} \right).2 \\
 & \equiv \text{comp}^{r \rightarrow s} \left(\underbrace{\lambda_z. B \left(\text{hcomp}^{r \rightarrow z} u_{0.1} \begin{bmatrix} \varphi & \hookrightarrow & u.1 \\ \varphi^* & \hookrightarrow & \lambda_z.u_{0.1} \end{bmatrix} \right)}_{L :=} \right) u_{0.2} \begin{bmatrix} \varphi & \hookrightarrow & u.2 \\ \varphi^* & \hookrightarrow & \lambda_z.u_{0.2} \end{bmatrix} \\
 & \equiv \text{hcomp}^{r \rightarrow s} (L s) (\text{coe}^{r \rightarrow s} L u_{0.2}) \begin{bmatrix} \varphi & \hookrightarrow & \lambda_z. \text{coe}^{z \rightarrow s} L (u z).2 \\ \varphi^* & \hookrightarrow & \lambda_z. \text{coe}^{z \rightarrow s} L u_{0.2} \end{bmatrix}
 \end{aligned}$$

- In general: $\lambda_z. \text{coe}^{z \rightarrow s} L u_{0.2}$ not equal to constant path at base

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