

# Progress Report on Constructive Higher Presheaf Models of HoTT

Thierry Coquand   Jonas Höfer   Christian Sattler

University of Gothenburg and Chalmers University of Technology

HoTT/UF 2025, Genoa, Italy

April 15, 2025

# Motivation

- We work in a *constructive* meta theory
  - Weak meta theory to make the results more generally applicable
  - Allows internalizing model construction into arbitrary models of ETT
- We want presheaf models for HoTT (over an internal category in **cSet**)
- Justification of Synthetic Approaches
  - Synthetic Algebraic Geometry (Cherubini, Coquand, and Hutzler, 2023)
  - Synthetic Stone Duality (Cherubini, Coquand, Geerligs, and Moeneclaey, 2024)
  - ...

## Why do we need an Internal Site?

### Validity of Duality Axioms

- $k\text{-Alg}_{fp} \rightarrow \mathbf{cSet} \cong (\square^{\text{op}} \times k\text{-Alg}_{fp}) \rightarrow \mathbf{Set}$  where  $\square$  is some cube category
- The generic ring is  $R(A) := k\text{-Alg}_{fp}(k[X], A) \cong |A|$
- The duality axiom says that for each presentation<sup>1</sup>  $(p_1, \dots, p_m): R[X_1, \dots, X_n]^m$

$$\begin{array}{ccc}
 \frac{R[X_1, \dots, X_n]}{(p_1, \dots, p_m)} & \xrightarrow{\text{ev}} & R^{(s: R^n) \times \forall_i. p_i(s) \simeq 0} \\
 & \searrow \eta & \uparrow \wr \\
 & & \frac{R[X_1, \dots, X_n]}{(p_1, \dots, p_m)} \text{ strict}
 \end{array}$$

- Since  $\simeq$  in  $R$  is extensional equality,  $R^{(s: R^n) \times \forall_i. p_i(s) \simeq 0}$  has no non-trivial paths.
- Strict axiom holds in general  $\implies$  holds for HITs iff quotients are equivalent

## Internal Sites

- Resolve mismatch by passing to *internal site* of *cubical (0-truncated)  $k$ -algebras*  
 $k\text{-Alg}: \square^{\text{op}} \rightarrow \mathbf{Cat}$     “The type of 0-truncated  $k$ -algebras in  $\text{Psh}(\square)$ ”
- This category is fully defined in the language of HoTT
- We need a strict category for what follows, but path composition is not strict!
  
- Use *relational* morphisms, i.e., phrase conditions for  $h: A \rightarrow B$  as  

$$(x, y, z: A) \rightarrow x + y \simeq_A z \rightarrow hx + hy \simeq_B hz$$
- For arbitrary  $\mathbb{C}$  analogous construction: consider image of  $\mathfrak{y}: \mathbb{C} \hookrightarrow \text{Psh}(\mathbb{C})$  and write natural transformations relationally  $uf \simeq v \rightarrow (\alpha_I u)f \simeq \alpha_J(v)$

## Internal Sites

- Category of internal presheaves on  $\mathbb{C}: \square^{\text{op}} \rightarrow \mathbf{Cat}$  is equivalent to  $\text{Psh}(\int_{\square} \mathbb{C})$
- In internal language of  $\text{Psh}(\square)$ : just an ordinary presheaf category

$$\text{Psh}(\square) \begin{array}{c} \xrightarrow{\pi^*} \\ \perp \\ \xleftarrow{\pi_*} \end{array} \text{Psh}(\int_{\square} \mathbb{C})$$

- The category  $\text{Psh}(\int_{\square} \mathbb{C})$  is a setting for the cubical model construction
  - tiny interval  $(\pi^*\mathbb{I})(I, X) = \mathbb{I}(I)$
  - cofibration classifier  $(\pi^*\Phi)(I, X) = \Phi(I)$
- We obtain a model of HoTT with HITs

## Working with Strictified Internal Sites

- Some key objects can be defined directly, e.g.,  $R(A) := |A|$ 
  - These will in general only be fibrant *levelwise*
- Other objects are more problematic, e.g., quotients of and free algebras over  $R$ 
  - These can be defined levelwise as an HIT
  - Are only presheaves *up to homotopy*

$$\begin{array}{ccc}
 \begin{array}{ccc}
 A & & \\
 \downarrow fg & \searrow g & \\
 & & B \\
 & \swarrow f & \\
 C & & 
 \end{array}
 & \longmapsto &
 \begin{array}{ccc}
 A[X] & & \\
 \downarrow \overline{fg} & \swarrow \overline{g} & \\
 & & B[X] \\
 & \swarrow \overline{f} & \\
 C[X] & & 
 \end{array}
 \end{array}$$

## Strictification Modality (Coquand, Ruch, and Sattler, 2021)

- Consider external categories, but these results can be generalized to our setting
- Introduce modality  $\underline{E}$  s.t.  $A \xrightarrow{\text{weak}} B$  correspond to  $A \xrightarrow{\text{strict}} \underline{E}B$

### Lemma (Levelwise Principle)

For an  $\underline{E}$ -modal type  $A$ , we have  $\text{El}_{\text{Psh}(f \mathbb{C})}(\Gamma, \|A\|) \longleftrightarrow \text{El}_{\text{Psh}(\square)}(\mathbb{C}_0.\Gamma, \|A\|)$ .

- The construction can be factored over a notion of weak presheaf
- Allows working with weak objects needed for SAG and SSD settings

$$\text{Psh}_w(f \Gamma) \begin{array}{c} \xrightarrow{R_\Gamma} \\ \text{“}\top\text{”} \\ \xleftarrow{U_\Gamma} \end{array} \text{Psh}(f \Gamma) \Big) \underline{E} \quad (\text{Nat in } \Gamma : \text{Psh}(\mathbb{C}))$$

## Summary until now

- To obtain the duality axioms, we need to pass to internal sites
- To deal with the resulting coherence issues, we need the modality  $\underline{E}$
- For axioms of *synthetic stone duality*, we also need dependent choice

## Sattler's Model of $\infty$ -Groupoids (2023)

- Let  $\square := \mathbf{Pos}_{\text{Fin}, \neq \emptyset}$  and  $\boxtimes: \text{Psh}(\square) \xrightarrow{i^*} \text{Psh}(\Delta_+) \xrightarrow{i_*} \text{Psh}(\square)$
- Defining property  $\text{El}_{\text{Psh}(\square)}(\Gamma, \boxtimes A) \cong \text{El}_{\text{Psh}(\Delta_+)}(i^* \Gamma, i^* A)$
- We take the submodel of modal types for  $\boxtimes$
- Associated model structure for this model is Quillen equivalent to the (constructive) Kan model structure on  $\text{Psh}(\Delta)$

### Lemma (Pointwise Principle)

For  $\boxtimes$ -modal types, there is a logical equivalence  $\text{El}_{\text{Psh}(\square)}(\Gamma, \|A\|) \longleftrightarrow \text{El}_{\mathbf{Set}}(|\Gamma|, |A|)$ .

- As a consequence, dependent choice holds in the model
- The presheaf construction should preserve this property

# Cubical Presheaves over Internal Categories

## Putting it all Together

### Lemma

*The pointwise lifted  $\square$  modality preserves fibrant types in the sense of  $\text{Psh}(f \mathbb{C})$ .*

To combine the two modalities and obtain both principles we need the following.

### Lemma

*The modality  $\underline{E}$  preserves  $\square$ -modal types.*

We **cannot** yet make the conclusion below. The underlying family of types of an internal presheaf might not be fibrant in the sense of  $\text{Psh}(\square)$ .

$$\text{El}_{\text{Psh}(f \mathbb{C})}(\Gamma, \|A\|) \xleftarrow{\underline{E}} \text{El}_{\text{Psh}(\square)}(\mathbb{C}_0 \cdot \Gamma, \|A\|) \xleftarrow{\square} \text{El}_{\text{Set}}(|\mathbb{C}_0| \cdot |\Gamma|, |A|)$$

# Cubical Presheaves over Internal Categories

## Fibrant Categories

**Issue**  $A \in \mathbf{Ty}_{\text{Psh}(f\mathbb{C})}(\Gamma)$  fibrant  $\not\Rightarrow$  associated  $A \in \mathbf{Ty}_{\text{Psh}(\square)}(\mathbb{C}_0.\Gamma, A)$  fibrant

### Definition

A cubical category  $\mathbb{C}: \square^{\text{op}} \rightarrow \mathbf{Cat}$  is *fibrant* if  $\mathbb{C}_1 \rightarrow \mathbb{C}_0 \times \mathbb{C}_0$  is a fibration.

- Equivalently,  $\mathbb{C}_1: \mathbb{C}_0 \times \mathbb{C}_0 \rightarrow \mathbf{U}$  family of fibrant types
- Given a path  $x: x_0 \simeq_{\mathbb{C}_0} x_1$  we can built a line  $f: (i: \mathbb{I}) \rightarrow \mathbb{C}_1(x_0, x_i)$  with  $f_0 = \text{id}_{x_0}$

### Lemma

Let  $\mathbb{C}: \square^{\text{op}} \rightarrow \mathbf{Cat}$  be a fibrant category.

If  $A \in \mathbf{Ty}_{\text{Psh}(f\mathbb{C})}(\Gamma)$  fibrant then the associated  $A \in \mathbf{Ty}_{\text{Psh}(\square)}(\mathbb{C}_0.\Gamma, A)$  is fibrant.

## Choice Axioms for Presheaf Models

### Theorem

For a fibrant cubical category  $\mathbb{C}: \square^{\text{op}} \rightarrow \mathbf{Cat}$ , the operation  $\underline{E} \circ \square$  is a lex modality on  $\text{Psh}(\int_{\square} \mathbb{C})$ , and the submodel of modal types is

1. a model of HoTT (with HITs),
2. with a logical equivalence  $\text{El}_{\text{Psh}(\int_{\square} \mathbb{C})}(\Gamma, \|A\|) \leftrightarrow \text{El}_{\mathbf{Set}}(|\mathbb{C}_0| \cdot |\Gamma|, |A|)$  natural in  $\Gamma$ .

- General tool for constructing presheaves models
- All our categories of interest satisfy the fibrancy condition
- The pointwise principle allows us to conclude dependent choice for SSD

## Conclusion and Future Work

- Finish specific applications (e.g. SAG, SSD, ...)
- Compare cubical with other models of higher presheaves
- Formalize existing construction
  - large parts can be done in internal language of  $\mathbf{Psh}(\square)$
  - especially the technical verification of axioms for the applications

## Dependent Choice Axiom from Pointwise Principle

- $\text{El}_{\text{Psh}(f \mathbb{C})}(\Gamma, \|A\|) \leftrightarrow \text{El}_{\text{Set}}(|\mathbb{C}_0|.|\Gamma|, |A|)$  implies dependent choice

$$\text{El}_{\text{Psh}(f \Gamma)}(\Gamma, \|A_0\|) \longrightarrow \text{El}_{\text{Set}}(|\mathbb{C}_0|.|\Gamma|, |A_0|)$$

$$\text{El}_{\text{Psh}(f \Gamma)}(\Gamma, \Pi_{n: \mathbb{N}, a_n: A_n} \|\text{Fib}(f_n, a_n)\|) \longrightarrow \text{El}_{\text{Set}}(|\mathbb{C}_0|.|\Gamma|, \Pi_{n: \mathbb{N}, a_n: |A_n|} \Sigma_{a_{n+1}: |A_{n+1}|} |f_n a_{n+1} \simeq a_n|)$$

- Then we can argue using induction in **Set** and conclude

$$\text{El}_{\text{Psh}(f \Gamma)}(\Gamma, \|\Sigma_{u: \Pi_{\mathbb{N}} A} \Pi_{i: \mathbb{N}} f_i u_{i+1} \simeq u_i\|) \longleftarrow \text{El}_{\text{Set}}(|\mathbb{C}_0|.|\Gamma|, \Sigma_{u: \Pi_{\mathbb{N}} |A|} \Pi_{i: \mathbb{N}} |f_i u_{i+1} \simeq u_i|)$$

## Cubical Presheaves over Internal Categories

**Setting**  $\square^{\text{op}}: \mathbb{C} \rightarrow \mathbf{Cat}$ , a strict (closed) category in  $\text{Psh}(\square)$

- The category  $\text{Psh}(\int_{\square} \mathbb{C})$  has
  - tiny interval  $(\pi^* \mathbb{I})(I, X) = \mathbb{I}(I)$
  - universal cofibration  $\pi^* \top: \pi^* \mathbf{1} \rightarrow \pi^* \Phi$
- Cubical model construction applies

### Problem

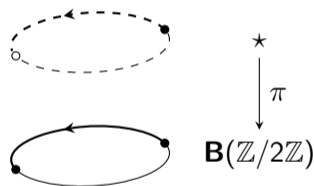
- If  $\mathbb{C} = \mathbb{C}_0$  the model is not the slice model!
- This is not the same issue as in CRS since it occurs for  $\mathbb{C}_0$
- There are non-trivial paths in the type of objects

$$\begin{array}{c}
 \text{Psh}(\square) \downarrow \mathbb{C}_0 \\
 \wr \\
 \text{Psh}(\int_{\square} \mathbb{C}_0) \\
 \begin{array}{c} \uparrow \quad \downarrow \\ i^* \quad \dashv \quad i_* \end{array} \\
 \text{Psh}(\int_{\square} \mathbb{C}) \\
 \begin{array}{c} \uparrow \quad \downarrow \\ \pi^* \quad \dashv \quad \pi_* \end{array} \\
 \text{Psh}(\square)
 \end{array}$$

# Cubical Presheaves over Internal Categories

## Notions of Fibrancy

- Comparison of fibrancy in case of **discrete** category
- Consider the family  $(- = \star)$  over  $\mathbf{B}(\mathbb{Z}/2\mathbb{Z})$ 
  - Fibrant in  $\text{Psh}(f \mathbf{B}(\mathbb{Z}/2\mathbb{Z}))$  constructed with  $\pi^*\mathbb{I}, \pi^*\Phi$
  - Modal for  $\Box$ -modality (lifted in the obvious way)
  - Inhabited on points since  $\star = \star$
  - **not inhabited**,  $\text{El}_{\text{Psh}(f \mathbf{B}(\mathbb{Z}/2\mathbb{Z}))}(1, (- = \star)) = \emptyset$
- This type is **not** fibrant in the slice






## Fibrancy

- An internal (dependent) presheaf is fibrant if  $A_0 \rightarrow \Gamma_0$  is a fibration, and the restriction action (the square on the left) is a morphism of fibrations

$$\begin{array}{ccccc}
 \mathbb{C}_0.\Gamma_0.A_0 & \longleftarrow & \mathbb{C}_1.\Gamma_1.A_1 & \longrightarrow & \mathbb{C}_0.\Gamma_0.A_0 \\
 \downarrow & & \downarrow & \lrcorner & \downarrow \\
 & \text{Mor} & & & \\
 \mathbb{C}_0.\Gamma_0 & \longleftarrow & \mathbb{C}_1.\Gamma_1 & \longrightarrow & \mathbb{C}_0.\Gamma_0
 \end{array}$$

- In the model we obtain from the construction, these will not be fibrations in the sense of  $\text{Psh}(\square)$
- Instead, we only consider those lifting problems where the path is constant on  $\mathbb{C}_0$

# Bibliography I

-  Cherubini, Felix, Thierry Coquand, and Matthias Hutzler (2023). *A Foundation for Synthetic Algebraic Geometry*. arXiv: 2307.00073 [math.AG]. URL: <https://arxiv.org/abs/2307.00073>.
-  Cherubini, Felix et al. (2024). *A Foundation for Synthetic Stone Duality*. arXiv: 2412.03203 [math.LO]. URL: <https://arxiv.org/abs/2412.03203>.
-  Coquand, Thierry, Fabian Ruch, and Christian Sattler (2021). “Constructive sheaf models of type theory”. In: *Mathematical Structures in Computer Science* 31.9, pp. 979–1002. ISSN: 0960-1295,1469-8072. DOI: 10.1017/S0960129521000359. URL: <https://doi.org/10.1017/S0960129521000359>.